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K. MIYA

Nuclear Engineering Research Laboratory
Faculty of Engineering, University of Tokyo, Japan

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Faster Eddy Current Computation Using Voronoi–Delaunay Transformation

Y. SAITO, S. IKEGUCHI and S. HAYANO

College of Engineering, Hosei University,
3–7–2 Kajinocho Koganei, Tokyo 184, Japan

ABSTRACT

Previously, we have proposed a locally orthogonal discretization method based on a Voronoi-Delaunay diagram for calculating the electromagnetic fields in a most efficient manner (Saito *et al.*, 1986, 1988, 1989). This locally orthogonal discretization method has been compeled to solve the two-independent systems (Voronoi and Delaunay). However,in this paper, we exploit the Voronoi-Delaunay transformation method that the solution of Delaunay system can be obtained by transforming the solution of Voronoi system. This Voronoi-Delaunay transformation method is now applied to the transient eddy current problems. As a result, it is found that the eddy current problems can be solved by means of the Voronoi-Delaunay transformation in an extremely efficient manner.

KEYWORDS

Eddy current; Field computation; Voronoi-Delaunay;

INTRODUCTION

In order to evaluate the electromagnetic fields in a most efficient manner, a geometrical duality between the Delaunay triangles and associated Voronoi polygons has been utilized to implement a dual energy finite element approach (Saito *et al.*, 1986, 1988, 1989). This method requires the use of a single potential to establish the upper and lower bounds of solutions, whereas the traditional dual energy finite element approach requires the use of two different types of potentials (vector and scalar) (Penman *et al.*, 1982, Hammond *et al.*, 1976, 1983). Therefore it is obvious that this method is able to provide the improved functional as well as improved local solution, while the traditional dual energy method is possible to provide only the improved functional. Even if a single type of potential is required to implement the dual energy approach, this method has been compeled to solve the two-independent systems(Voronoi and Delaunay).

In this paper, we exploit the Voronoi-Delaunay transformation method that the solution of Delaunay system can be obtained by transforming the solution of Voronoi system. This means that only the Voronoi system of equations has to be solved to implement the dual energy approach. This Voronoi-Delaunay transformation method is now applied to the transient eddy current problems. As a result, it is found that the eddy current problems can be solved by means of the Voronoi-Delaunay transformation in an extremely efficient manner.

VORONOI-DELAUNAY TRANSFORMETION METHOD

Basic field equation

In two dimensional x-y plane, most of the eddy current problems is reduced to solve a following equation

$$\frac{1}{\mu}\frac{\partial^2 A}{\partial x^2}+\frac{1}{\mu}\frac{\partial^2 A}{\partial y^2}-\sigma\frac{\partial A}{\partial t}=-J_s , \tag{1}$$

where A, J_s, μ and σ are the z-component of vector potential, source current density, permeability and conductivity, respectively. The vector potential A is related with the flux density B by

$$\nabla \times A = B \quad , \tag{2}$$

so that the electric field intensity E is given by

$$E = -\nabla \phi - \frac{\partial A}{\partial t} \quad . \tag{3}$$

The scalar potential ϕ and time derivative term in (3) are respectively related with the source current density J_s and eddy current density J_e as

$$J_s = -\sigma \nabla \phi \quad , \tag{4}$$

$$J_e = \sigma \ (\partial A / \partial t) \quad . \tag{5}$$

Voronoi-Delaunay diagram

Delaunay triangulation of arbitrary set of points is constructed by considering the properties of its geometrical dual i.e., the set of Voronoi polygons. Delaunay triangles are related to Voronoi polygons in that the circumcenters of Delaunay triangles are the vertices of the Voronoi polygons. Figure 1 shows the triangles in Delaunay mesh, and the Voronoi polygons associated with these Delaunay triangles are shown by dashed lines.

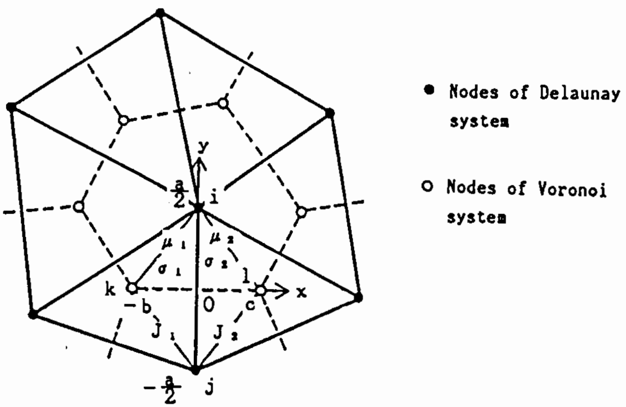


Fig.1.Voronoi-Delaunay diagram and a locally orthogonal coordinate system.

Locally orthogonal form

By considering Fig.1, it is obvious that the Delaunay triangles and Voronoi polygons are locally orthogonal : each triangle side is perpendicular to the corresponding Voronoi polygon edge. When we select a set of nodal variables A_i, A_j, A_K, A_l, then a following interpolating function may be assumed :

$$A = a_0 + a_1 x + a_2 y + a_3 x y \quad . \tag{6}$$

However, it is difficult to represent the coefficients a_0, a_1, a_2, a_3 in terms of the nodal variables A_i, A_j, A_K, A_l, uniquely. This means that two complete but independent sets of nodal variables must be defined : one is located at the vertices of the Delaunay triangles ; and the other is located at the vertices of the Voronoi polygons. A simple Lagrange interpolation between the nodes i and j in Fig.1 yields a trial function for the Delaunay system as

$$A = (1/2)(A_i + A_j) + (A_i - A_j)(y/a) \ , \tag{7}$$

where a is the distance between the nodes i and j (*Saito* et al., 1986, 1988, 1989). When we assume that each of the Delaunay triangles takes a distinct permeability μ, then a flux density B_x $[=\partial A/\partial y]$ must be continuous to both regions 1 and 2 in Fig.1. Equation (7) satisfies this boundary condition automatically. On the other side, a field intensity H_y $[=-(1/\mu)\partial A/\partial x]$ must be common to both regions 1 and 2 in Fig.1. This boundary condition can be satisfied by the following interporating functions between the nodes k and l in Fig.1 :

$$A = \{(A_k/\mu_1 b) + (A_l/\mu_2 c) + (A_l - A_k)x/(\mu_2 b c)\} / \{(1/\mu_1 b) + (1/\mu_2 c)\}, \quad -b \leqq x \leqq 0 \quad , \tag{8a}$$

$$A = \{(A_k/\mu_1 b) + (A_l/\mu_2 c) + (A_l - A_k)x/(\mu_1 b c)\} / \{(1/\mu_1 b) + (1/\mu_2 c)\}, \quad 0 \leqq x \leqq c \quad , \tag{8b}$$

where the distances b and c are shown in Fig.1 (Saito *et al.*, 1986, 1988, 1989).

According to these two independent interpolating functions (7) and (8a) or (8b), the governing equation (1) may be reduced to a one-dimensional equation in either the Delaunay or in the Voronoi sets of variables :

$$\frac{1}{\mu}\frac{\partial^2 A}{\partial x^2} = \frac{1}{2}\left[\sigma\frac{\partial A}{\partial t} - J_s\right], \tag{9}$$

$$\frac{1}{\mu}\frac{\partial^2 A}{\partial y^2} = \frac{1}{2}\left[\sigma\frac{\partial A}{\partial t} - J_s\right]. \tag{10}$$

Functionals and nodal equations

A functional which automatically satisfies the B_x $[=\partial A/\partial y]$ condition between the adjacent Delaunay triangles is given by

$$F(A) = \iint\{(1/\mu)(\partial A/\partial y)^2 + [\sigma(\partial A/\partial t) - J_s]A\}dxdy\ . \tag{11}$$

After substituting Eq.(7) into Eq.(11) and integrating over the region enclosed by i-k-j-l in Fig.1, we can obtain the functional $F(A)$ for the Delaunay system. By taking an extremum of this functional $F(A)$, a nodal equation for the node i in Fig.1 can be obtained as

$$\left(\frac{1}{\mu_1}\frac{b}{a} + \frac{1}{\mu_2}\frac{c}{a}\right)(A_i - A_j) + \frac{a}{24}(b\sigma_1 + c\sigma_2) \times \frac{\partial}{\partial t}(7A_i + 5A_j) = \frac{a}{4}(bJ_1 + cJ_2)\ . \tag{12}$$

Entire Delaunay system of equations is represented by

$$D_D\Phi_D + E_D(d/dt)\Phi_D = F_D\ , \tag{13}$$

where D_D , E_D are the coefficient matrices corresponding to the first and second terms on the left of Eq.(12) ; F_D is an input source current vector corresponding to the right of Eq.(12) ; and Φ_D is the potential vector of Delaunay system, respectively.

On the other side, a functional which satisfied the H_y $[=-(1/\mu)\partial A/\partial x]$ condition between the adjacent Delaunay triangles is given by

$$G(A) = -\iint\{\mu(1/\mu)(\partial A/\partial X)^2 + [\sigma(\partial \hat{A}/\partial t) - J_s]\hat{A}\}dxdy\ . \tag{14}$$

where ^ refers to the prescribed values.

After substituting Eqs.(8a) and (8b) into Eq.(14) and integrating over the region enclosing by i-k-j-l in Fig.1, we can obtain the functional G(A) for the Voronoi system. By taking an extremum of this functional G(A), a nodal equation for the node k in Fig.1 can obtained as

$$[1/(\mu_1 \frac{b}{a}+\mu_2 \frac{c}{a})](A_k - A_l) + \frac{ab}{2}\sigma_1 \frac{\partial}{\partial t} A_k = \frac{ab}{2} J_1 . \quad (15)$$

Entire Voronoi system of equations is represented by

$$D_V \Phi_V + E_V (d/dt) \Phi_V = F_V , \quad (16)$$

where D_V, E_V are the coefficient matrices corresponding to the first and second terms on the left of Eq.(15) ; F_V is an input source current vector corresponding to the right term of Eq.(15) ; and Φ_V is the potential vector of Voronoi system, respectively.

Voronoi-Delaunay Transformation

The Voronoi system of equations has been derived satisfying with two boundary condition of tangential field intensity H_t between two adjacent Delaunay triangles. However, as shown in Fig.2(a), it is obvious that the normal components flux density B_n to the edges of Delaunay triangle are included in the entires solution of Voronoi system. The nodal variables which satisfy the boundary condition of normal flux density B_n are essentially located at the vertices of Delaunay triangle. A governing equation which must be satisfied by these nodal variables is given by

$$\frac{\partial}{\partial x}\left(\frac{1}{\mu}\right)\frac{\partial A}{\partial x} + \frac{\partial}{\partial y}\left(\frac{1}{\mu}\right)\frac{\partial A}{\partial y} = 0 , \quad (17)$$

where the current densities J_s and J_e in Eq.(1) have been lumped to the vertices of Voronoi polygons. Hence, by means of the Delaunay discretization (12), the vector potential A_i in Fig.2(b) can be represented in terms of the vector potentials of Voronoi system as

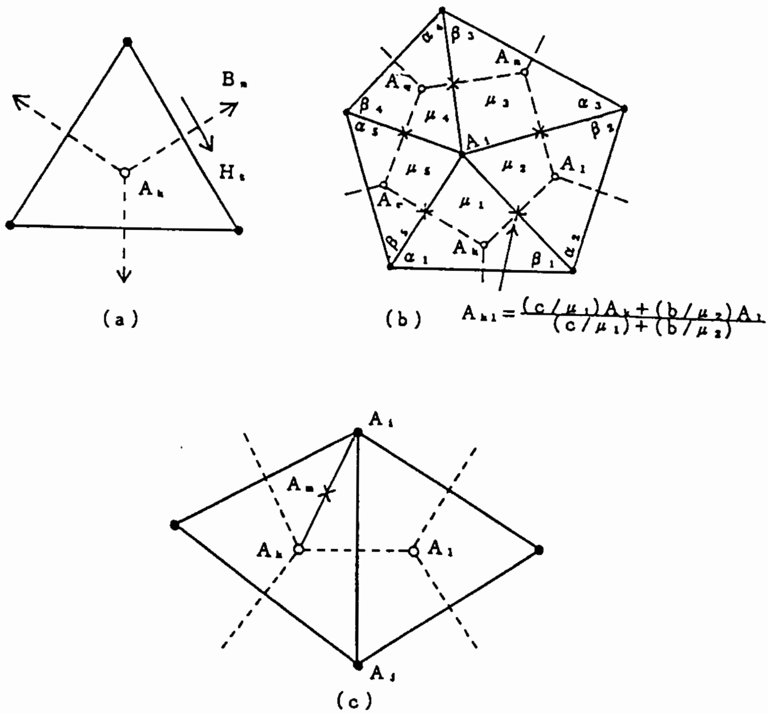
$$[(1/\mu_1)\cot\alpha_1+(1/\mu_2)\cot\beta_2]A_{k1}+[(1/\mu_2)\cot\alpha_2$$

$$+(1/\mu_3)\cot\beta_3]A_{1n}+[(1/\mu_3)\cot\alpha_3+(1/\mu_4)\cot\beta_4]A_{nq}$$

$$+[(1/\mu_4)\cot\alpha_4+(1/\mu_5)\cot\beta_5]A_{qr}+[(1/\mu_5)\cot\alpha_5$$

$$+(1/\mu_1)\cot\beta_1]A_{kr}=[\sum_{j=1}^{5}(1/\mu_j)(\cot\alpha_j+\cot\beta_j)]A_i, \qquad (18)$$

where the angles $\alpha_1 \sim \alpha_5$, $\beta_1 \sim \beta_5$ are shown in Fig.2(b) ; and A_{k1}, A_{1n}, A_{nq}, A_{qr}, A_{kr} are the potential at the intersected points of Voronoi polygon and Delaunay triangle edges in Fig.2(b). For example A_{k1} is



$$A_{k1}=\frac{(c/\mu_1)A_k+(b/\mu_2)A_l}{(c/\mu_1)+(b/\mu_2)}$$

Fig.2. (a) The normal component of flux density B_n and tangential component of fielg intensity H_t.
(b) Transformation from the Voronoi nodal variables to Delaunay nodal variables.
(c) Location of the mid point potential A_m.

$$A_{kl}=[(c/\mu_1)A_k+(b/\mu_2)A_l]/[(c/\mu_1)+(b/\mu_2)]. \tag{19}$$

Thus, by means of Eq.(19), the potential vector Φ_D of the Delaunay system can be represented in terms of the connection matrix C and the potential vector Φ_V as

$$\Phi_D = C\Phi_V . \tag{20}$$

By means of Eq.(20), the Delaunay system of equations (13) is transformed into the Voronoi system by

$$C^T D_D C\Phi_V + C^T E_D C(d/dt)\Phi_V = C^T F_D , \tag{21}$$

where a superscript T refers to the transposed matrices. Thereby, a total Voronoi system of equation becomes

$$D\Phi_V + E(d/dt)\Phi_V = F, \tag{22}$$

where

$$D = (1/2)[C^T D_D C + D_V] , \tag{23a}$$

$$E = (1/2)[C^T E_D C + E_V] , \tag{23b}$$

$$F = (1/2)[C^T F_D + F_V] . \tag{23c}$$

A coefficient (1/2) in Eqs.(23a)-(23c) is required because a simple summation of Eqs.(16) and (20) duplicaties the input source current vector. Each of the transient solution vector Φ_D, Φ_V in Eqs.(13) and (16) exhibits a different behaiviour depending on their system eigen-values, but the solution vector Φ_V in Eq.(22) exhibits their averaged behaiviour. This means that a promising transient responce of Eq.(1) may be computed by Eq.(22) even if a small number of nodes is employed. Further improvement of the solutions is possible when we consider the potentials located at the mid points between the vertices of Voronoi polygon and Delaunay triangle. The mid point potential vector Φ_M can be represented in terms of the vectors Φ_V and Φ_D as

$$\Phi_M = C_{DM}\Phi_D + C_{VM}\Phi_V , \tag{24}$$

where C_{DM}, C_{VM} are the interpolating matrices between the vertices of

Voronoi polygon and Delaunay triangle. For example, a mid point potential A_m in Fig.2(c) in given by

$$A_m = (1/2) A_i + (1/2) A_k \ . \tag{25}$$

By means of Eq.(20), Eq.(24) is reduced into

$$\Phi_M = (C_{DM} C + C_{VM}) \Phi_V \ . \tag{26}$$

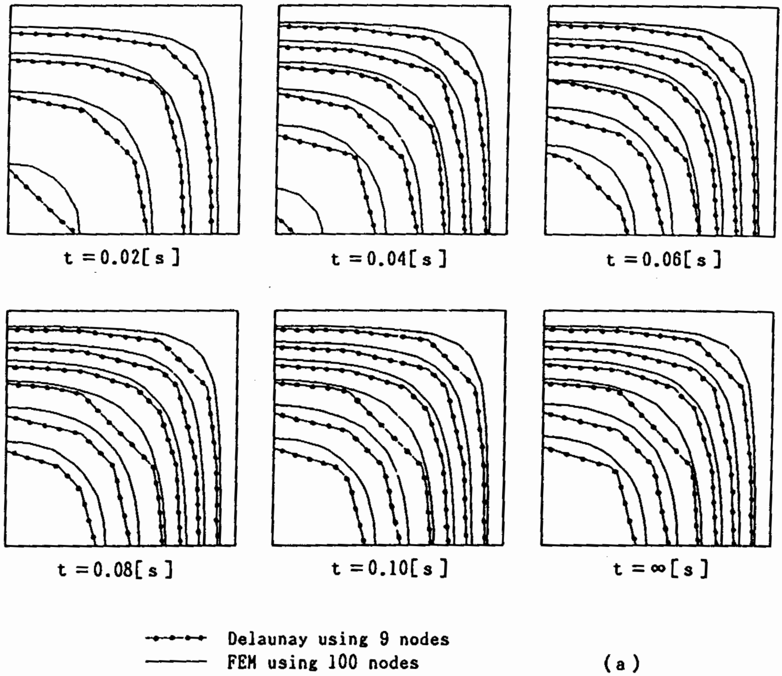
The potential vector Φ_M in Eq.(26) is obviously improved in accuracy because the vector Φ_M takes into account the boundary condition both of the tangential field intensity H_t and normal flux density B_n in Fig.2(a).

An example

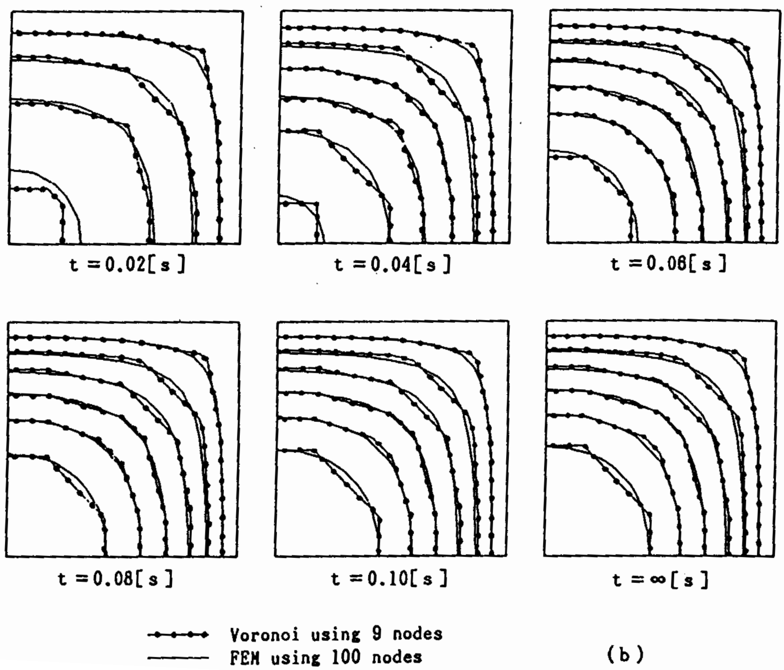
The method is illustrated by applying to a dynamic magnetic field calculation of ferromagnetic material with square cross section (Saito *et al.*, 1986, 1988, 1989). The time discretization of Eq.(21) was carried out by the conventional trapezoidal method. Various constants used in the calculations are listed in Table 1. Figures 3(a) and 3(b) show the transient field distributions computed by the Delaunay (13) and Voronoi (16) systems, respectively. Figure 3(c) shows the results of computations obtained by the Voronoi-Delaunay transformation of Eqs.(22) and (26). For comparison, the solution computed by the standard first order triangular finite element method is also shown in these figures. By considering Figs.3(a)-3(c), it is obvious that our Voronoi-Delaunay transformation method provides an excellent result even if a small number of nodes is employed.

TABLE 1. Various constants used in the computations.

Permeability μ	2.513[H/m]
Conductivity σ	20[1/Ω-m]
Step source current density J_s	2[A/m²]
Step width in time Δt	0.001[sec]


t =0.02[s]
t =0.04[s]
t =0.06[s]
t =0.08[s]
t =0.10[s]
t =∞[s]
Delaunay using 9 nodes
FEM using 100 nodes

(a)


t =0.02[s]
t =0.04[s]
t =0.06[s]
t =0.08[s]
t =0.10[s]
t =∞[s]
Voronoi using 9 nodes
FEM using 100 nodes

(b)

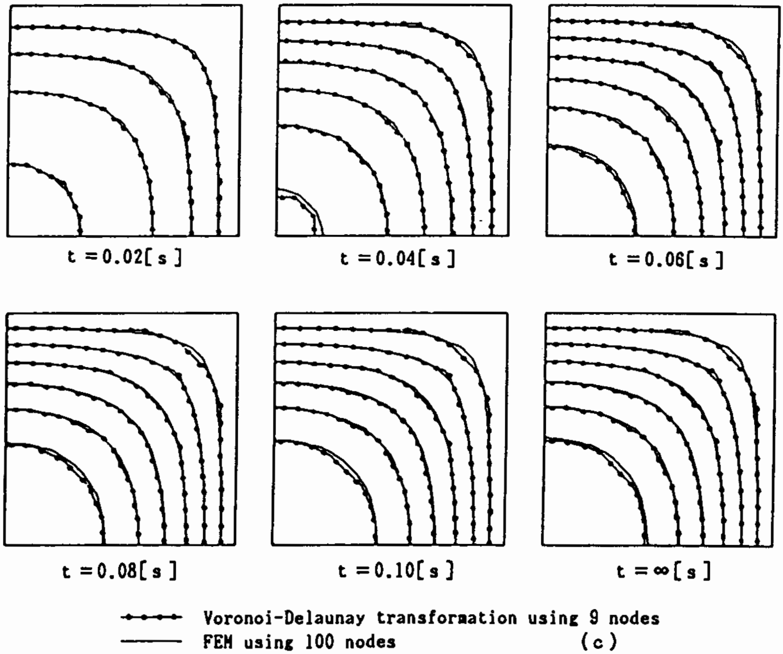


Fig.3. Transient field distributions computed by (a) the Delaunay (b) Voronoi (c) Voronoi-Delaunay transformation methods together with the standard first order triangular finite element method.

CONCLUSION

Previously proposed locally orthogonal discretization method was a quite effective method to calculate the electromagnetic fields in an efficient manner, but it was compeled to solve the two independent systems i.e., Voronoi and Delaunay. To overcome this defficiency, we have proposed, in this paper, the Voronoi-Delaunay transformation method, which requires only one system of equations to implement a dual energy approach. As a result, it is revealed that the transient eddy current problems can be solved by our new method in an extremely efficient manner.

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