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Wavelet Solution of The Inverse Parameter Problems

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Abstract – **Previously, we have proposed a method of solving inverse problems, and successfully applied the method to biomagnetic fields as well as the nondestructive testing in metallic materials. In the present article, we propose a novel inverse approach for the parameter determination problems employing wavelet analysis. A simple example of parameter determination demonstrates the validity of our wavelet approach.**

I. INTRODUCTION

Inverse problems are classified into two major categories, i.e. one is the inverse parameter problem; the other is the inverse source problem. For the inverse parameter problem, it is possible to obtain a unique solution if the fields are measured ideally; such as medium parameter identification in human body employing the computed tomography (CT). However, most of the inverse problems are generally reduced to solving a system equation for which it is difficult to obtain a unique solution. In order to overcome this difficulty, we have previously proposed a method of solving the inverse problems, and successfully applied it to biomagnetic fields as well as to nondestructive testing in metallic materials [1,2].

On the other hand, the wavelet analysis has been studied for image data compression and analyzing the spectrum of image in informatics [3-6].

In the present article, we propose a novel approach for the inverse parameter problems employing wavelet analysis. The key idea is that a system matrix of the inverse problems is regarded as two-dimensional image data. The two-dimensional wavelet transform is applied to this system matrix. An approximate inverse matrix of the system is obtained from the wavelet spectrum. We here consider a test example in which the relationship between input and output is evaluated from given input and output data. As a result, the example demonstrates the validity of our wavelet approach.

II. DISCRETE WAVELET TRANSFORM

A. One-dimensional wavelet transform

In the present paper, we employ Haar's analyzing wavelets [3]. Let us consider a following linear transformation

$$\mathbf{X}' = C\mathbf{X}, \tag{1}$$

where $\mathbf{X}$ is a data vector with order n; n must be a power of 2; and C is

$$C = \begin{bmatrix} c_0 & c_1 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ c_1 & -c_0 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & 0 & c_0 & c_1 & \cdot & \cdot & 0 & 0 \\ 0 & 0 & c_1 & -c_0 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & c_0 & c_1 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & c_1 & -c_0 \end{bmatrix}. \tag{2}$$

In equation (2), the first, third, fifth, and the other odd rows generate the components of data convolved with the coefficients c_0, c_1. This corresponds to a weighted integral operation. On the other hand, the even rows generate the components of data convolved with the coefficients $c_1, -c_0$. This corresponds to a weighted differential operation [3,5].

In order to carry out an inverse transformation, the coefficients c_0, c_1 should be determined by a relationship:

$$C^T C = I, \tag{3}$$

where I is a n-th order unit matrix and a superscript T refers to the transpose of matrix C.

From equations (2) and (3), we have

$$c_0^2 + c_1^2 = 1. \tag{4}$$

Equation (4) has two unknowns c_0, c_1, but we have only one equation. To determine the coefficients c_0, c_1, generally, a following conditions is considered:

$$c_0 - c_1 = 0. \tag{5}$$

From equations (4) and (5), we have

$$c_0 = \frac{1}{\sqrt{2}}, \qquad c_1 = \frac{1}{\sqrt{2}}. \tag{6}$$

The pair of coefficient c_0, c_1 in (6) is Haar's analyzing wavelets, which are equivalent to Daubechies's second order

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analyzing wavelets [3,4].

For simplicity, let us consider a data vector $\mathbf{X}$ with order 8:

$$\mathbf{X} = [x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7 \quad x_8]^T. \tag{7}$$

Applying the transform matrix C_8 to (7) yields

$$\mathbf{X}' = C_8\mathbf{X} = [s_1 \quad d_1 \quad s_2 \quad d_2 \quad s_3 \quad d_3 \quad s_4 \quad d_4]^T. \tag{8}$$

The elements in vector $\mathbf{X}'$ are sorted by using the following matrix:

$$P_8 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{9}$$

Thus, we have

$$P_8\mathbf{X} = P_8C_8\mathbf{X} = [s_1 \quad s_2 \quad s_3 \quad s_4 \quad d_1 \quad d_2 \quad d_3 \quad d_4]^T. \tag{10}$$

Further transformation to the elements s_1, s_2, s_3, s_4 in (10) yields

$$W^{(2)}\mathbf{X} = [S_1 \quad S_2 \quad D_1 \quad D_2 \quad d_1 \quad d_2 \quad d_3 \quad d_4]^T, \tag{11}$$

where

$$W^{(2)} = (P_8'C_8')(P_8C_8), \quad P_8' = \begin{bmatrix} P_4 & 0 \\ 0 & I_4 \end{bmatrix}, \quad C_8' = \begin{bmatrix} C_4 & 0 \\ 0 & I_4 \end{bmatrix}. \tag{12}$$

Similar transformation to S_1, S_2, D_1, D_2 in (11) yields

$$W^{(3)}\mathbf{X} = [\mathbf{S}_1 \quad \mathbf{D}_1 \quad D_1 \quad D_2 \quad d_1 \quad d_2 \quad d_3 \quad d_4]^T. \tag{13}$$

where

$$\begin{aligned} W^{(3)} &= (P_8''C_8'')(P_8'C_8')(P_8C_8), \\ P_8'' &= \begin{bmatrix} P_2 & 0 \\ 0 & I_6 \end{bmatrix}, \quad C_8'' = \begin{bmatrix} C_2 & 0 \\ 0 & I_6 \end{bmatrix}. \end{aligned} \tag{14}$$

The wavelet transform of one-dimensional data with order 8 is finally given by $3 (= Log_2 8)$ steps of linear transformation. Equation (13) is the finally obtained wavelet spectrum. The elements $\mathbf{S}_1, \mathbf{D}_1$ in (13) are called the Mother Wavelet coefficients, and the others are called the wavelet coefficients at each level.

Inverse wavelet transform is carried out by

$$\begin{aligned} \mathbf{X} &= [W^{(3)}]^T (W^{(3)}\mathbf{X}), \\ [W^{(3)}]^T &= [(P_8''C_8'')(P_8'C_8')(P_8C_8)]^T, \\ &= (P_8C_8)^T (P_8'C_8')^T (P_8''C_8'')^T, \\ &= C_8^T P_8^T (C_8')^T (P_8')^T (C_8'')^T (P_8'')^T. \end{aligned} \tag{15}$$

B. Two-dimensional wavelet transform

The discrete wavelet transform can be extended to two dimensions [3]. Usually, two-dimensional wavelet transform is applied to a square matrix. In this article, two-dimensional wavelet transform is generalized to a rectangular matrix. The generalized two-dimensional wavelet transform is given by

$$M' = W_n M W_m^T, \tag{16}$$

where M' and M are the transformed (spectrum) matrix and original matrix with order n by m, respectively. W_n and W_m are the wavelet transform matrices with order n by n and m by m, respectively.

The inverse wavelet transform is carried out by the following equation:

$$M = W_m^T M' W_n. \tag{17}$$

III. THE INVERSE PARAMETER PROBLEMS

A. Wavelet approach

The key idea is that the system matrix is regarded as one of the image data. The system matrix as an image data is transformed into a space of wavelet spectrum.

Let us consider an inverse parameter problem. The system $\mathbf{X} = C\mathbf{Y}$ can be modified by exchanging the elements in the vector $\mathbf{Y}$ and matrix C, viz.

$$\mathbf{X} = Y\mathbf{C}, \tag{18}$$

where a matrix Y and vector $\mathbf{C}$ are the system matrix composed of the elements in $\mathbf{Y}$ and parameter vector to be determined, respectively. In order to solve for (18), we apply the two-dimensional discrete wavelet transform to (18). The system matrix Y is transformed by

$$Y' = W_n Y W_m^T. \tag{19}$$

From the result of (19), it is found that the spectrum matrix can be classified into two major groups. One group takes the large absolute value, and the other takes the smaller absolute value. We take a square matrix S around the Mother wavelet coefficient out of the entire wavelet spectrum Y'. Generally, the square matrix S around the Mother wavelet coefficients have large values. After taking of the inverse matrix of S, we embed this inverse matrix into a zero matrix $\mathbf{Z}$ with order m by n.

$$Y'^{-1}_{Appro} = S^{-1} \to \mathbf{Z}. \tag{20}$$

Equation (20) means that the inverse matrix S^{-1} is embedded at the top square region of $\mathbf{Z}$.

The approximate inverse matrix Y^{-1}_{Appro} of the system is obtained by the two-dimensional inverse wavelet transform:

$$Y^{-1}_{Appro} = W_n^T Y'^{-1}_{Appro} W_m. \tag{21}$$

Finally, the parameter vector of system $\mathbf{C}$ is given by

$$\mathbf{C} = Y^{-1}_{Appro}\mathbf{X}. \tag{22}$$

Thus, the parameter vector $\mathbf{C}$ can be obtained from the known input Y and output $\mathbf{X}$.

B. An example

Let us consider an example of parameter identification problems. For this example, the current and magnetic field distribution are known vectors, but the relationship between them is unknown. This example is reduced to solving for the following system equation

$$\mathbf{X} = Y\mathbf{C}, \tag{23a}$$

or

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 & \cdots & y_n & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdots & 0 & y_1 & \cdots & y_n & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & y_1 & \cdots & y_n \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \\ \vdots \\ \vdots \\ c_{mn} \end{bmatrix}, \tag{23b}$$

where $\mathbf{C}, \mathbf{X}$ and Y are a vector of system parameter to be determined, an output vector, and the system matrix composed of the input current, respectively.

Figure 1 shows an example of a parameter identification problem from both input currents and output magnetic field vectors. Actually, exact parameter of the vector $\mathbf{C}$ in (23b) are determined by the Ampere's law. We verify that the exact parameter can be identified by the wavelet approach.

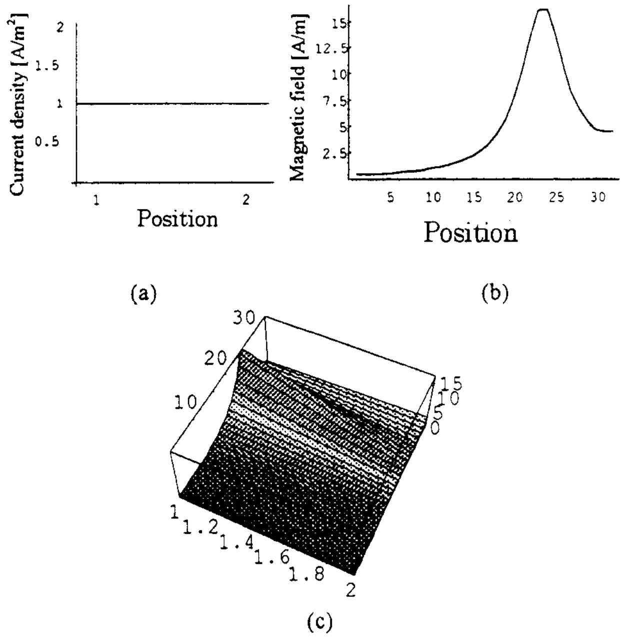

Fig.1. (a) An input current vector $\mathbf{Y}$, (b) an output magnetic field vector $\mathbf{X}$, and (c) the system matrix.

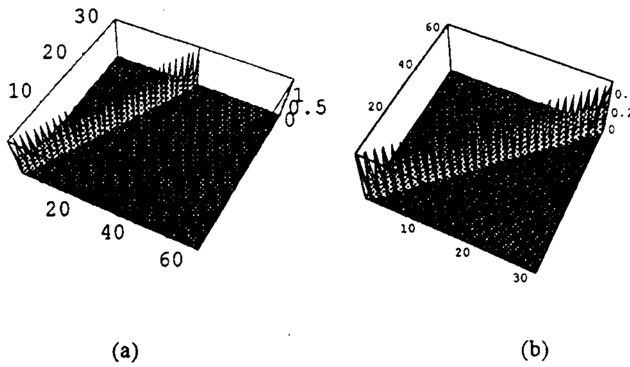

Fig. 2. (a) Two-dimensional wavelet spectrum Y' of the system, and (b) an approximate inverse matrix Y^{-1}_{Appro}.

Figures 1(a), 1(b) and 1(c) show an input current distribution, an output magnetic field distribution, and the system matrix of this parameter identification problem in (23b), respectively.

Figures 2(a) and 2(b) show a two-dimensional wavelet spectrum Y' of the system in figure 1(c) and an approximate inverse matrix Y^{-1}_{Appro} of the system, respectively.

Finally, the parameter vector $\mathbf{C}$ in (23) is given by

$$\mathbf{C} = Y^{-1}_{Appro}\mathbf{X}. \tag{24}$$

Figures 3(a) and 3(b) show the determined parameter of the system, and reproduced magnetic field distribution, respectively. The result in 3(a) coincides with those of the Ampere's law. Actually, the parameter of the system is determined by the Ampere's law. Thus, we have succeeded

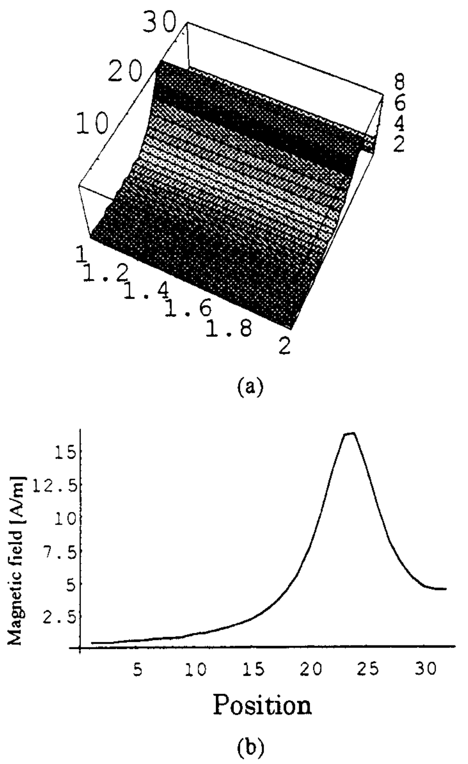

Fig.3. (a) The decided system matrix, and (b) the reproduced magnetic field distribution by the estimation system matrix.

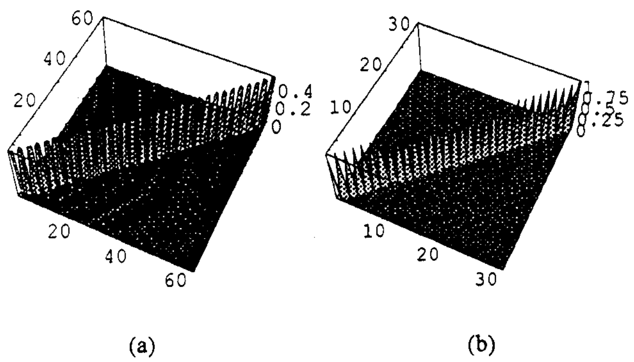

Fig.4. (a) The left-inverse matrix check $Y_{Appro}^{-1}Y$ and (b) the right-inverse matrix check YY_{Appro}^{-1} .

in estimating the parameter of the system from both the input and output vectors.

C. Validity of the approximate inverse matrix

Mathematical validity of the inverse matrix is generally carried out by means of the left- and right-inverse matrix checks. In this inverse parameter problem, the left-inverse matrix check $Y_{Appro}^{-1}Y$ is not equivalent to the right-inverse matrix check YY_{Appro}^{-1}, because the system matrix is rectangular. When the left-inverse matrix check $Y_{Appro}^{-1}Y$ becomes

$$Y_{Appro}^{-1}Y = \mathbf{I}_m, \tag{25}$$

the solution vector can be uniquely determined, where $\mathbf{I}_m$ is an identity matrix with order m.

When the right-inverse matrix check YY_{Appro}^{-1} becomes

$$YY_{Appro}^{-1} = \mathbf{I}_n, \tag{26}$$

the existence of solution vector can be confirmed, where $\mathbf{I}_n$ is an identity matrix with order n.

Thus, the left-inverse matrix check means the uniqueness of solution. The left-inverse matrix check shown in figure 4(a) is similar to the identity matrix $\mathbf{I}_m$. This means that an approximate solution vector could be expected. Also, the right-inverse matrix check shown in figure 4(b) is the identity matrix $\mathbf{I}_n$.This means that the existence of solution vector could be expected.

IV. CONCLUSION

In the present paper, we have proposed a novel approach for the inverse parameter problems employing the wavelet analysis. The wavelet analysis is applied to the system matrix of the inverse parameter problems. The results reveal that our wavelet approach is possible to get an approximate inverse matrix of the system. A simple example has demonstrated the validity of our approach.

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